EQUALS, EXPRESSIONS, EQUATIONS, AND THE MEANING OF VARIABLE: A TEACHING EXPERIMENT

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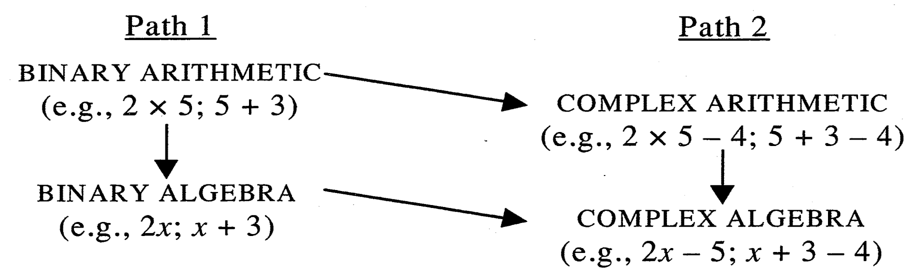
A teaching experiment was designed to facilitate Year 8 students' understanding of algebraic expressions and equations through the use of unknowns, patterns, relationships, and concrete materials. This paper discusses the study's theoretical framework, the teaching episodes relating to expressions, equations, and equals, the students' reactions to this instruction, and their initial and final understandings of algebraic expressions, equations, and equals. Exploration of the successful and unsuccessful teaching episodes emphasises the relationship between instruction, prior knowledge and learning.

Algebra is an abstract system in which components interact to reflect the structure of arithmetic. Understanding algebraic expressions requires abstract schema (Ohlsson, 1993) of the arithmetic *operational laws* and *equals*, combined with the algebraic notion of *variable*. For example, the distributive principle holds for whole numbers [e.g., 2×(3+4)=(2×3)+(2×4)], decimal numbers (e.g., 4×4.7=4×4+4×0.7) and mixed numbers (e.g., $5\times3^2/_{11}=5\times3+5\times^2/_{11}$), and it also holds for algebra [e.g., $4(x+y)=4x+4y$]. Thus, the distributive principle is an isomorphic structure between arithmetic and algebra. It is an abstract schema because its meaning lies in terms of relationships [a(b+c)=ab+ac], not the particular content (e.g., fractions).

Difficulties in learning algebra have long been documented (e.g., Thorndike et al., 1923) and more recent research (e.g., Boulton-Lewis et al., 1998; Linchevski & Herscovics, 1996) continues to show that achievement rates in algebra are poor. Research indicates that instruction does not seem to be bridging the gap between arithmetic and algebra, particularly in: (a) developing meaning for variables (Booth, 1988; Cooper at al., 1997; Linchevski & Herscovics, 1996) and for the equals sign (Herscovics & Linchevski, 1994); (b) connecting the knowledge required to solve arithmetical equations by inverting or undoing (backtracking), and the knowledge required to solve algebraic equations by operating on or with the unknown (Booth, 1988; Herscovics & Linchevski, 1994); (c) overcoming the syntactic similarity between the algebraic notation for $3x$ and the arithmetic notation for 2-digit place value (Stacey & MacGregor, 1997); and (d) abstracting the properties and conventions of operations (Herscovics & Linchevski, 1994). In response to the poor achievement rates, instructional practices which focus on patterns and physical materials (e.g., cups to represent variables, counters to represent numbers, balance beam for equals) to introduce algebra have been developed (e.g., Quinlan et al., 1993). However, patterns may not be effective as they do not easily lead to the generalisations required for algebraic understanding (Boulton-Lewis et al., 1997; MacGregor & Stacey, 1995). The use of physical materials may impose additional cognitive demands (Halford & Boulton-Lewis, 1992), contain intrinsic restrictions (Behr, Lesh, Post, & Silver, 1983), and have limited connections to symbols (Boulton-Lewis et al, 1998; Hart, 1989).

To overcome the above problems, Boulton-Lewis et al. (1997) proposed a *two-path* instructional model (see Figure 1). The model was based on the belief that understanding of complex algebra is the end product of a learning sequence of mathematical concepts that includes: *binary arithmetic*; *complex arithmetic* (a series of operations on numbers); and *binary algebra*. It means that 2x5 and 5+3 (binary operations) are a prerequisite for $2x$ and x+3 (binary algebra) while, in turn, 2x5-4 and 5+3-4 (complex arithmetic) forms an important prerequisite to understanding $2x$-4 and x+3-4 (complex algebra). It also means that understanding operational laws should be applied to series of operations as well as individual operations, and that learning complex algebra is facilitated by understanding similar (isomorphic) structures in complex arithmetic.

Figure 1
Two-path Model (Boulton-Lewis et al., 1997) for Algebra Instruction



Sometimes it is difficult to classify activities into the four areas of Figure 1 (e.g., equations with one unknown which can be solved by *backtracking*, a method from complex arithmetic). Thus, instruction to develop algebraic knowledge should be seen as encompassing three stages, namely, arithmetic through pre-algebra (where arithmetic techniques are used with letters, e.g., $3(x+2)=18$) to algebra (where operations act on variables, e.g., $x+7=3x-1$) (Boulton-Lewis, Cooper, Atweh, Pillay, & Wilss, submitted).

THE STUDY

The study was a teaching experiment (Romberg, 1992) and an intervention design (Hiebert & Wearne, 1991) undertaken with 51 Year 8 students (two classes) at a middle-class suburban state secondary school. Twenty 40-minute episodes (separated into two two-week units, beginning two months apart) to introduce early algebra were taught to one class and repeated with the second. One of the researchers did the teaching whilst the class teacher and another researcher observed. Each teaching episode was videotaped, all students' work was collected, all students were surveyed at regular intervals (including the beginning of the teaching episodes), and a representative sample of 14 students (7 from each class) was interviewed after each unit. The monitoring of student responses and reactions facilitated the modification of succeeding teaching episodes, and permitted the study of the relationship between teacher actions and student learning.

The major purpose of the teaching episodes was to have students reflect on their experience of arithmetic in order to draw out generalities (e.g., those that differentiate between addition, subtraction, multiplication and division and those that underlie the procedures used in simplifications and equation solving in algebra). For the purposes of this paper, the planned sequence of teaching episodes in the study was as defined below. The teaching episodes were informed by the findings of Kaplan, Yammamoto and Ginsburg (1989) and Linchevski & Herscovics (1996) in that they aimed to underpin formal mathematical knowledge with informal mathematical notions, particularly with respect to the repeated addition notion of multiplication (e.g., $3x=x+x+x$).

1. *Review of the four operations and equals.* This aimed at highlighting the structure of the operations, as discussed at the beginning of this paper.

2. *Introduction to the notions of expressions (e.g., 34+58) and equations (e.g., 4+5=9, 34+28=31x2).* This also aimed at emphasising the legitimate changes that can be made to expressions (doing and undoing so they stay the same value) and equations (do the same to both sides so the sides are equivalent). For example, expressions such as 24+58 remain the

same value if multiplied by 10 and then divided by 10, or increased by 15 and then decreased by 12 and by 3. Equations such as 25+11=36 are equivalent if both sides are multiplied by 17, i.e., (25+11)x17=36x17.

3. *Development of the notion of variable.* This was done through Usiskin's (1988) three approaches of unknowns (e.g., $3x=6$), patterns (e.g., 3, 7, 11, 15, ...) and relationships (e.g., $2 \rightarrow 5$, $8 \rightarrow 23$, $5 \rightarrow 14$), and through materials (e.g., cups and counters). Usiskin's (1988) approaches were introduced within complex arithmetic (see Figure 1) and then revisited in turn with concrete materials to introduce variable. Unknowns were taught first (because of their pre-algebraic focus), this teaching based on the transformational approach to arithmetic (e.g., 2x3 is viewed as a transformation from 2 to 6 through multiplying by 3) (Cooper & Baturo, 1992). This method has been shown to be effective in teaching complex arithmetic and introducing *backtracking* (see Figure 2), the method most students use to solve algebraic equations such as $3x+4=19$ (Boulton-Lewis et al., 1998; Herscovics & Linchevski, 1994).

Figure 2
Transformations and Unknowns

$4 \rightarrow \times 3 \rightarrow 12 \rightarrow +2 \rightarrow 14$	Transformation
$? \rightarrow \times 3 \rightarrow ? \rightarrow +2 \rightarrow 20$	Unknown
$6 \leftarrow /3 \leftarrow 18 \leftarrow -2 \leftarrow 20$	Backtracking
$x \rightarrow \times 3 \rightarrow 3x \rightarrow +2 \rightarrow 3x+2$	Variable as unknown

For patterns, a sequence of numbers was to be generated and related to the numbers of the term (directly, or by using a geometrical construction, e.g., the number of matchsticks needed to make 5 squares in a row). The nature of the *n*th term would be discussed. For relationships, numbers would be related to other numbers (either directly, or through a "guess my rule" game) and generalisation sought for the relationships. The ways in which the three approaches are related to variable are highlighted in Figure 3 below. Binary and complex examples are illustrated. Modelling of the expressions was to be undertaken in both directions; the teacher directing the students to show an expression with cups and counters, and the teacher modelling expressions with cups and counters with students giving the expressions.

Figure 3
Three Approaches for Introducing Variable

UNKNOWNS	PATTERNS	RELATIONSHIPS
$3 \rightarrow \times 3 \rightarrow 9$ $11 \rightarrow \times 3 \rightarrow 33$ $? \rightarrow \times 3 \rightarrow 21$ $x \rightarrow \times 3 \rightarrow 3x = 21$	Term: 1, 2, 3, 4, . . . , n Value: 3, 6, 9, 12, . . ., $3n$	$\times 3$ $8 \rightarrow 24$ $4 \rightarrow 12$ $13 \rightarrow 39$ $y \rightarrow 3y$
$3 \rightarrow \times 3 \rightarrow 9 \rightarrow +2 \rightarrow 11$ $11 \rightarrow \times 3 \rightarrow 33 \rightarrow +2 \rightarrow 35$ $? \rightarrow \times 3 \rightarrow 21 \rightarrow +2 \rightarrow 23$ $x \rightarrow \times 3 \rightarrow 3x \rightarrow +2 \rightarrow 3x+2 = 23$	Term: 1, 2, 3, 4, . . . , n Value: 5, 8, 11, 14, . . ., $3n+2$	$\times 3 + 2$ $8 \rightarrow 26$ $4 \rightarrow 14$ $13 \rightarrow 41$ $y \rightarrow 3y+2$

4. *Extension of variable to more complex expressions [e.g., 3x+2; 3(x+2)] and equations (e.g., 3x+2=11).* As each of the approaches moved towards *variable*, cups and counters were to be used to model expressions of the form $3x$ and $x+3$ (binary algebra) and $3x+2$ and $3(x+2)$ (complex algebra).

Fourteen students were surveyed at the beginning of the intervention regarding their understanding of equals, expression $2x+3$, and equation $3x-4=11$. The same students were interviewed at the end of the intervention regarding equals, expressions $3x$, $3x+1$ and $3(x+2)$, and equation $3x-4=11$. This interview also focused on students' ability to represent the expressions and equations with materials and to relate them to patterns.

RESULTS AND DISCUSSION

The students' responses for the pre-survey and post-interview are presented first, followed by the results of the teaching episodes.

Survey and Interview Responses

Table 1 summarises the fourteen students' responses to the initial survey tasks and the final interview tasks. The discussion focuses on the intervention's effectiveness for developing understanding of equals, expressions, and equations.

Table 1
Students' Appropriate Initial and Final Responses to Items Associated with Equals, Expressions, and Equations (n=14)

Appropriate Initial Responses	No.	Appropriate Final Responses	No.
Equals		*Equals*	
Meaning of the sign "="	3	Meaning of the sign "="	6
Expressions		*Expressions*	
Meaning of 2x+3	1	Meaning of 3x	13
		Meaning of $3x+1$	12
		Meaning of $3(x+2)$	12
Equations		*Equations*	
Meaning of $3x-4=11$	1	Meaning of $3x-4=11$	11

At the beginning of the teaching episodes, three students could interpret the equals sign as "the same as" or "equivalence"; others were concerned with getting an answer. (These results concur with previous findings by Behr, Erlwanger, and Nichols, 1992, and Cooper et al., 1997.) Only one student was able to explain the meaning of an algebraic expression or equation. Inappropriate responses were associated with getting an answer (expressions only), doing something (e.g., replacing x), or incorrectly interpreting the coefficient of x (e.g., twenty something for $2x$). Over half the students offered no responses, suggesting a low understanding of, or lack of familiarity with, algebraic expressions and equations at the start of the study.

As Table 1 also shows, there was an improvement in students' explanations of the "=" sign after the intervention. However, over 50% of students gave no response or an inappropriate response. Contrary to this, there was a clear improvement in students' ability to explain the meaning of algebraic expressions such as $3x$, $3x+2$, and $3(x+2)$. In particular, students were comfortable with the meaning of variable as "any number," and there was no confusion with place value (thirty something for $3x$). However, explanations involving multiplication and grouping were more common than those involving repeated addition. Most students providing appropriate explanations could also represent the first two expressions with cups and counters. Less than one third of the students could correctly use them to represent the third expression, $3(x+2)$. Pattern construction was also found to be poor.

At the end of the teaching episodes, the majority of the 14 students were able to find an expression and an equation, and use real world examples. Fewer students could provide an adequate rationale for their choice of expression or equation. The greatest difficulties seemed to be associated with the allowable changes that can be made, particularly with respect to expressions.

Teaching Episodes

To address the problems highlighted in the previous section, the teaching episodes emphasised the notion of equals as equivalence (e.g., 36÷9 is the same value as 2x2), and focused on the differences between expressions (no equals, containing operations) and equations (equals, operations on one or both sides of equals). The episodes also developed the notion, somewhat informally, that equals "balances" (mathematically) both sides of the equation. The equals lessons appeared to be successful, the students particularly enjoyed doing the worksheets (which were in a game environment) associated with the lessons, and they completed them correctly. However, the expression and equation lessons were less successful. Discussions with groups of students during the worksheet activity indicated that there was little understanding of what was expected, nor of the differences between expressions and equations. To overcome this, the introduction to transformations (including *backtracking*) was brought forward to help students with the idea of doing and undoing arithmetic operations. The second attempt at teaching the difference between expressions and equations appeared to be more successful, and students seemed to experience no difficulties with the transformation worksheets. However, observations of the worksheet activity on expressions and equations continued to show that a significant proportion of the classes exhibited little understanding of the difference. An additional worksheet was designed to relate them to real world instances, as shown in Figure 4 below. These worksheets were completed eagerly and correctly.

Figure 4
Expressions and Equations

Expression	5x4	There were 5 packets with 4 chocolates in each packet
Sum	5x4=	There were 5 packets with 4 chocolates in each packet – how many chocolates?
Equation	5x4=20	There were 5 packets with 4 chocolates in each packet – this made 20 chocolates.
Equation	5x4=18+2	There were 5 packets with 4 chocolates in each packet – this was the same number of chocolates as John ,who originally had 18 and was given 2 more.

Observations of teaching episodes and analysis of worksheets relating to introducing variable indicated that the students' responses to the episodes were mixed. The transformational episodes were again well received in that students appeared to like and understand the teaching of transformations with unknowns. In fact, the interview students commented that the transformational activities were their best lessons, and their favourite worksheets were those concerned with equals, transformations and relating expressions and equations to real life situations. The visual representation of change as an arrow seemed to assist in understanding the effect of operations on variables, particularly the difference between $3x+2$ and $3(x+2)$. The episodes dealing with patterns and relationships were not as well received, and students appeared to have difficulty determining the generalisations for the patterns and relationships, particularly when there were two operations. Hence, the teaching episodes were modified to include hints (e.g., the hint "multiply by 2" was placed beside relationships, $3 \rightarrow 7$, $12 \rightarrow 25$, $5 \rightarrow 11$, $a \rightarrow ?$).

The episodes in which cups and counters were used to introduce variable and to represent algebraic expressions were also not as successful as expected. The students tended to model $3x$ and $x+3$ with the same material (3 counters for the 3 and a cup for the x) and, when this was overcome, to repeat the error for 3x+2 and $3(x+2)$ (3 cups and 2 counters for both expressions). They appeared not to like using the materials. This was reinforced by the interview, where 9 of the 14 students felt that the cups and counters were a negative part of the teaching. Only 3 of the 14 students liked any of the worksheets concerned with introducing variables; this was equally the case for unknowns as with patterns and relationships.

The students' responses with the cups and counters showed that they did not have a facility with multiplication in terms of repeated addition. (This phenomenon may have been a consequence of the focus on the *array* model of multiplication, rather than on the *repeated addition* model, in Queensland schools). Therefore, extra time was scheduled in the next episode to teach understanding of multiplication as repeated addition, and representing $3x$ as $x+x+x$ (similar to Linchevski & Herscovics, 1996). After this, the teaching episodes on variable, including representation with cups and counters, appeared to be more successful. According to the observers, the students' responses during these teaching episodes indicated that most students were grasping the meanings being portrayed.

Overall, a difficulty with sequencing emerged. The episodes were designed so that all the complex arithmetic work in unknowns, patterns and relationships was completed before variables were introduced using these approaches. When the episodes returned to each approach for the development of variable, the students appeared to have forgotten the approach and the arithmetic activities had to be repeated. As a consequence of this, later episodes were designed so that arithmetic work led straight on to variable work for each particular activity.

CONCLUSIONS

Several main conclusions could be made following the results of the study.

1. *Arithmetic as a basis for algebra*. The interaction between the understanding of arithmetic and the understanding of algebra was more complex than expected. The students' reactions to the teaching episodes revealed that reflecting on arithmetic to build algebra generally worked, but only if the arithmetic lead straight to the algebra generalisations for each activity. The two path model (see Figure 1) appeared to provide a framework for effective teaching. However, rather than consisting of four separate steps to be performed across time, it represents a framework that should be followed for each separate notion and principle.

2. *Equals*: While there was improvement in students' understanding of the "=" sign during the study, inadequate responses persisted. The correct meaning of equals appears to need ongoing reinforcement.

3. *Expressions and equations*. The episodes were successful in helping students to gain understanding of the meaning of expressions and equations. However, one problem often encountered in secondary school mathematics is the use of incorrect procedures in making changes to expressions and equations. Attempts to address this problem here were not overly successful. It may be that more complex understanding is required to understand operations an expressions and equations. The findings supported Linchevski and Herscovics (1996) in stressing the importance of understanding $3x$ as repeated addition. Although extra work was undertaken, repeated addition was still a weakness of the students in examples which included more than one operation (e.g., interpreting $3(x+2)$ as $(x+2)$ plus $(x+2)$ plus $(x+2)$). However, there appeared to be a relationship between understanding multiplication as repeated addition and correctly using cups and counters for multiplication situations such as $3(x+2)$.

4. *Usiskin's approaches*: Usiskin's approaches (e.g., unknowns, patterns and relationships) seemed a useful way to introduce algebra because of the way they reflect the different notions of generality. However, students had problems with patterns and relationships for more than one operation. Unknown as transformation was well received by the students and appeared in most cases to be understood. Therefore, as found by Herscovics and Linchevski (1994), variable as unknown and arithmetic transformation appear to offer good opportunities for the transition from arithmetic to algebra and seem to fit within the "pre-algebraic" level between arithmetic and algebra, as proposed by Boulton- Lewis et al, 1997.

5. *Cups and counters*: The use of materials (cups, counters) acted as a conduit between arithmetic and algebraic notions. They appeared to work successfully for $3x$ (binary algebra) and $3x+1$ (simple complex algebra). Although it appeared useful for students to differentiate between multiplication then addition (e.g., $3x+1$) and addition then multiplication [e.g., $3(x+2)$] with cups and counters, understanding was not evident in the final interview. This could reflect the criticisms of materials given by Halford and Boulton-Lewis (1992), Boulton-Lewis et al. (1998), and Hart (1989) that children see little connection between materials and symbols due to the additional cognitive demands that materials bring.

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